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Supersymmetric Lattice Models. Field Theory Correspondence, Integrabillity

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# Summary

Condensed matter is a discipline within physics that aims to understand the physical properties of a large variety of materials. The different phases in which these materials can exist, as well as the phase transitions between these phases, are studied. At a phase transition the physical properties of a material change drastically.

Besides the phases which we see every day, like solids, liquids and gases, there are numerous other phases, like the superconducting phase or the Bose-Einstein condensate. To accurately describe the particles, atoms and electrons making up a material, quantum mechanics is a necessary ingredient.

The physical properties of many materials are understandable using physical models. The basis of such a model is a Hamiltonian that describes the energy of the interactions between the different particles. Often the interactions between the particles are relatively weak, such that the system can be approximated by a system consisting of free particles without interactions. The interactions can then later be added, order by order, in a parameter that describes the interactions. This method results in a description in terms of quasi-particles, which are the effective particles originating from a combination of the real particles and their interactions with the environment.

This approach works for example very well for materials which become superconducting close to absolute zero temperature. However, materials have been discovered that become superconducting at far higher temperatures. For these materials the interactions between the electrons must be so strong that even the relatively high temperature (the highest temperature is around -135 °C at the moment) does not result in a high resistance in the material.

To understand such a material a theoretical model describing strong interactions is needed. A Hamiltonian for a system with strong interactions can be written down, but in many cases it is very hard to use it to calculate physical properties of the system. Many experiments aimed at understanding strongly correlated system are taking place, but there is also an important task for theoreticians. New methods should be developed to be able to say something about these types of systems. With these new methods it may in the future be possible to understand for example the mechanism that underlies high temperature superconductivity.

To understand strongly correlated systems it is useful to take a step away from the real materials and first study an idealised model in full detail. In this thesis we discuss a series of models that are called the $\mathrm{M}_k$ models and were originally introduced in ref. [6]. These models describe strongly interacting particles on a lattice, which we always take to be one-dimensional. The particles, which are fermions without spin, thus sit on a line. The interactions between them are such that in the $\mathrm{M}_k$ model at most $k$ particles can sit next to each other on the lattice points of the line. In chapter 2 of this thesis the $\mathrm{M}_k$ models and its properties are introduced.

Symmetries play an important role in physics. If a model is symmetric under a certain operation, it becomes easier to solve. Also phase transitions are described in terms of symmetries. For example, under the phase transition from a fluid to a solid, the continuous translation symmetry of the fluid gets broken to a discrete translation symmetry respecting the crystal structure of the solid. A special feature of the $M_k$ models is that they are defined in such a way that they have an unusual type of symmetry, so called supersymmetry. This symmetry was first introduced in the context of particle physics and relates two types of particles, bosons and fermions, to each other. In the $M_k$ lattice model this symmetry relates states with an odd and an even number of particles on the lattice to each other. Adding or removing a particle using this symmetry does not cost any energy. Supersymmetry can for example be used to find the number of ground states of the model. Because of the strong interactions between the particles there are several ways in which the particles can form a lowest energy state. In general determining the number of ground states for different lengths of the lattice would be difficult, but supersymmetry makes it an easy task.

One of the results of this thesis is that in the $M_2$ model, for the right choice of parameters, there are two extra so called dynamical supersymmetries. It then costs zero energy to add four lattice sites and three particles, or two lattice sites and one particle to the one dimensional chain. Moreover, for the same parameters the model turns out to be exactly solvable (integrable). In principle this makes it possible to calculate all energies of the system via what is called a Bethe ansatz. These results are described in chapter 3.

An important point in a phase transition is the critical point. At the phase transition between a liquid and a gas the critical point is characterised by one specific temperature and pressure at which the density of the gaseous phase and the liquid phase become equal and the two phases become indistinguishable. At this point the transition between liquid and gas becomes a continuous phase transition. Many other phase transitions are always continuous. In a magnet the size of the magnetic domains become smaller as the temperature increases, until, at the critical point, the fluctuations in the size of the domains become so large that no length scale can be recognized in the system anymore. The absence of a length scale characterizes the critical point and causes an extra symmetry in the system, called scale invariance.

In this thesis scale invariance plays an important role. The $M_k$ models describe a system with a certain length and number of particles. If the length and number of particles are very big, explicit calculations become impossible. But in this limit we can instead use another theory, a so called quantum field theory. In general it is hard to know what this quantum field theory exactly should look like. If a quantum field theory in two dimensions (space and time) has a scaling symmetry in addition to the standard symmetries of translations, rotations and Lorentz transformations, then this theory has a conformal symmetry and becomes a conformal field theory. In the case of the $M_k$ models there is also supersymmetry and the theory is a supersymmetric conformal field theory. Supersymmetric conformal field theories are extensively researched and classified. The theory that describes the critical point of the $M_k$ models is therefore precisely known. The relation between the $M_1$ model and the corresponding supersymmetric conformal field theory was studied in ref. [17]. In chapter 4 of this thesis we describe this relation in detail for the $M_2$ model and in later chapters we also look briefly into the $M_k$ models for higher values of $k$.

In chapter 6 of this thesis, the relation between the $M_k$ models and supersymmetric conformal field theories is made stronger. Instead of starting with the lattice model and comparing it to the field theory, we do exactly the opposite. We start with a supersymmetric conformal field theory and show that, with the right techniques, it is possible to cut finite pieces out of this theory. This 'finitised' theory leads to restrictions on a lattice model that is related to the field theory. In ref. [6] this technique was used to invent the $M_k$ model. In this thesis we make this relation between field theory and lattice model precise.

The critical point of the $M_k$ models is well understood due to the large amount of symmetry present. It is very interesting to also understand these models away from the critical point. For this the parameters for which the model is integrable play an important role. For the $M_2$ model these are described in chapter 3, and in ref. [9] they were found for all $M_k$ models. A particular choice of parameters gives the critical point. However, we can also periodically vary the parameters along the lattice points in a certain way, which is referred to as staggering. The model then stays integrable and goes to a massive phase, where there is a gap between the lowest energy state of the model and the higher energy states. In a quantum field theory this corresponds to a theory of massive particles (which has rest energy $mc^2$), which is why we call it a massive phase. In the critical phase, on the other hand, the energy difference between the lowest lying excited states and the ground states of the model goes to zero for large system lengths.

In chapter 5 we study what happens to the eigenstates of the $M_2$ and $M_3$ lattice model when we follow them from the critical point to the massive phase. To achieve this we make the staggering bigger and bigger until it is so big that at certain lattice sites it costs zero energy to add a particle. We also study specific defects in the lattice model. For example, in the $M_2$ model two particles are not allowed to sit next to each other at the location of a defect. Adding these types of defects leads to an interesting change in the relation with the conformal field theory and in the limit of extreme staggering extra ground states appear. The defects turn out to behave like quasi-particles with a fractional charge that satisfy the fusion rules of the fields in the corresponding conformal field theory. These fusion rules are rules that describe how two fields combine if they have an excitation at the same place.

There is also an interesting relation between one-dimensional $M_k$ lattice models and two-dimensional physical systems in which the Hall effect appears. The Hall effect is observed if a magnetic field is applied perpendicular to the direction of an electrical current. The magnetic field gives rise a voltage, the Hall voltage. At extremely low temperatures, plateaus arise in the value of the

Hall voltage and the voltage is quantised. This is the quantum Hall effect. The wavefunctions of the electrons in a quantum Hall system can be calculated via a conformal field theory. These conformal field theories are exactly the same as those that are related to the $M_k$ models. Relations between the $M_k$ models and quantum Hall systems can also be found in the massive phase. This is described in chapter 5.

Because we choose to study the massive phase for parameters for which the $M_2$ model is integrable, we again know enough about the system to identify the corresponding quantum field theory. In this case this is a massive, integrable, supersymmetric quantum field theory. In chapter 7 the first steps are made to make the relation between the massive $M_2$ model and this quantum field theory precise, by taking the limit of extreme staggering of the lattice model. In this limit the excitations of the $M_2$ model can be described as kinks in between three different ground states. These kinks correspond to fundamental excitations in the massive field theory.

The research in this thesis primarily contributes to a better understanding of the $M_k$ models and their relations with other models. With more research, these relations can likely be made more precise in the future. Because we use supersymmetry as a tool and restrict ourselves to integrable models, we do not directly describe any actual physical materials. However, the research into these models does contribute to a better understanding of strongly correlated particles, which in the future will be useful in studying phenomena in materials where strong interactions are important.