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Topological Phases in Condensed Matter Systems. A Study of Symmetries, Quasiparticles and Phase Transitionsality Pathology

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Summary

In the following pages I will give an overview of the field, which the work described in this thesis is part of. Towards the end, I will discuss specific details of my research, but I will mainly focus on the bigger picture here.

This summary is aimed at the reader who has no background in physics, but would like to know more about this topic. To prevent this text from becoming too long, which would do no justice to the word 'summary', I will refer to frames in the text with which I elaborate on certain concepts. These can be skipped to obtain a short version or can be examined by the reader who wants to see specific examples.

It is far from trivial to explain a topic in theoretical physics in such a way that it is comprehensible for a reader without any background in mathematics. I will do my best to succeed in this task as much as possible. At the same time a text without a reader is pointless and in my opinion the reader has an important role in this process too. Do not tell yourself in advance that you do not understand anything about the topic, but try to think of what it is exactly that seems unclear. Sometimes one must pass over certain uncertainties without worrying too much that the rest will be incomprehensible too ①.

> ① The fear of physics often stems from the underlying mathematics. Mathematics is like a language and therefore it is not surprising that it is not so easy to understand. Often you hear people claim that you are simply either good or bad at it. However I think that as with any language it takes a lot of practise to become skilled at it.
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> Physics, on the other hand, can be understood on a very different level. Compare it with the Uyghur people in China. We do not have to be familiar with Uyghur or Chinese to understand parts of the history of these peoples and their conflict.
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> In my opinion physics can also be understood at such a level. For example, there are different types of particles in the universe, and various forces are present. Some particles attract each other, others repel. If many particles come together, they may form specific materials and so on.

Theoretical physics Physics is the science which studies how nature behaves. Which types of particles exist and which forces act? The most obvious way to test something is by carrying out an experiment, and this branch of physics is called experimental physics. For example, we can measure the time it takes for a ball with a mass of 3 kg to roll down a certain incline. But what do these results tell us? We can only draw conclusions for this particular case. Should we be interested in the same experiment, but with a ball with a mass of 3.1 kg, we would have to repeat it.

> ② For example, when we consider classical mechanics, Newton's second law $\mathbf{F} = m\mathbf{a}$ tell us that if a force $\mathbf{F}$ acts on an object with mass m it will have an acceleration given by $\mathbf{a}$. One could claim that a physical phenomenon is now described by a mathematical formula, but that is not the case yet. We want to know how the object behaves no matter how far we go back in the past or how far we look forward in time. We know its acceleration, but what is its position and velocity?

(3) I want to highlight two examples, one in which the experimentalists led the way, and another in which the theorists were first.

In 1879, Edwin Hall conducted an experiment in which he discovered that a small voltage difference arises in a conductive material when an electric current flows in one direction, and a magnetic field is applied perpendicularly to the current. The voltage difference is observed in the direction which is perpendicular to both the current and the magnetic field. Nowadays this phenomenon is called the Hall effect and with our present knowledge the voltage difference can be explained as follows. Without a magnetic field, the electrons that make up the current would follow a straight path. When a magnetic field is applied the path gets curved by the Lorentz force and charge starts to accumulate at the edge of the sample, resulting in a voltage difference that is proportional to the magnetic field and the current. The explanation is fairly simple, but realize that Hall did this experiment almost 20 years before the electron was discovered.

As an example of a prediction that came from theoretical physics I would like to discuss the Majorana particle. In 1928 Paul Dirac formulated an equation which describes the electron within the theory of quantum mechanics. One implication of his theory was that antimatter should exist. Another was derived in 1937 by Ettore Majorana. He realized that a solution to the Dirac equation could be found which corresponds to a particle that is its own antiparticle, the so-called Majorana particle. Even though the theory allows for such a particle to exist that does not mean that it has been observed yet. Some physicists believe that the neutrino could be a Majorana particle. The neutrino is present in abundance in our universe, but it is also very difficult to detect. Another candidate can be found in the same realm of my field of research. Approximately 10 years ago it was realized that Majorana particles can be 'built' when certain materials are combined in a clever way. In 2012, a group in Delft conducted an experiment along those lines and they have strong evidence that they did indeed observe a Majorana particle, although I must say it has not yet been indisputably proven.

This is where theoretical physics enters the stage. By using mathematical structures (the language of nature), one tries to explain and predict the behavior of particles and forces. Writing down a theory is highly nontrivial. Simply using your imagination is not good enough, because the theory must be self-consistent, it should potentially fit within other existing theories and be reconciled with what is actually observed. (In this respect, the comparison that some people make between religion and science does not hold in my opinion.) And even if a theory can be written down that does not mean that it has been solved (2). Oftentimes the theory is too complicated if we want to take all the effects into account and a number of assumptions have to be made. For instance, that the presence of a butterfly in Brazil has no effect on an experiment with a particle accelerator in Texas.

Once more I would like to emphasize the power of the interaction between experimental and theoretical physics, as this is the main reason why the field in which my research took place is so appealing to me. Sometimes an experimental discovery occurs, which was unexpected and cannot be explained from theory (there are no formulas or models that describe this phenomenon). The community of theoretical physicists will try their very best to adjust the theoretical framework or come up with something new in order to explain the observation. On the other hand, a new theory may have implications which have not yet been observed experimentally. Experimentalists will design setups and perform measurements to test if the theory is indeed correct, or they may use the theory to design devices with practical applications for society (3). This interaction is a major driving force behind the acquisition of knowledge.

One could claim that the ultimate goal of physics is to construct one theory which accounts for everything we observe around us. Unfortunately this is not yet the case. When stretching the limits of our knowledge, scientists work in many different areas. As a result the different corners of physics drift away from each other. Occasionally a brilliant physicist (4) enters the stage and manages to connect two different theories, but it seems that a theory of everything will not be developed in the near future. Even so, not every physicist is concerned with building a theory of everything, and for me this was never the main goal. I have been working in one specific area and have been trying to add to the knowledge of it.

(4) I mentioned the Dirac equation in (3) mainly to introduce the Majorana particle, but the discovery of Paul Dirac is a good example of a theory which merges two previously different areas. He combined the theory of relativity of Albert Einstein with quantum mechanics in order to get a correct equation of motion for the electron.

The theory of relativity is necessary to describe objects that travel with a speed that approaches the speed of light ($= 300,000,000$ m/s). Quantum mechanics is a theory which was developed during the first decades of the last century by many different physicists. It describes physics when we go to a scale the size of an atom or even smaller.

In our daily lives we do not reach these speeds and we are definitely bigger than an atom, but that does not mean that the theory of relativity or quantum mechanics does not apply to us. It is just that classical mechanics is good enough and far more easy to work with, but for an electron classical mechanics breaks down and we have to appeal to the Dirac equation.

Condensed matter The area in which my research took place is called condensed matter theory. As opposed to high-energy physics which focuses in particular on the building blocks of the universe - the so-called elementary particles, such as the electron and for example the Higgs particle - condensed matter theory studies the interplay of particles when you bring many of them close together.

A common example is water. A glass of water and a block of ice both consist of many water molecules and yet these systems behave very differently: you can swim in water and sit on ice. Without having to know the exact position of all the molecules (which is impossible with these numbers of particles), and the forces which they exert on each other, the properties of the entire system can still be described. Water is an example of a liquid and ice an example of a solid. This is what we call different *phases* of matter. The phase a specific material is in depends on certain external features, such as the pressure or temperature of the system. As the temperature

(5) Superconductivity is a phenomenon in which some materials below a certain temperature (close to absolute zero) are in a phase in which the electric resistance through the material becomes equal to zero. It was discovered in 1911 in an experiment conducted by the Dutch scientist Heike Kamerlingh Onnes, for which he later received the Nobel Prize. Nowadays superconductivity is still an active area of research. Firstly, because the theory behind it is not entirely understood and secondly, because it has many applications in industry.

of a liquid decreases, there is a certain temperature at which the material changes from a liquid into a solid, which is referred to as a *phase transition*. I have used water as an example because we encounter it in our everyday lives and therefore it is easier to grasp, but at the same time it is limiting because there exist many other materials with far more exotic properties, for instance superconductivity (5).

Symmetry As mentioned before a material consists of a lot of particles and it is impossible to write a theory that keeps track of every separate particle and the kind of forces they exert on each other. Fortunately, theoretical physics does not only solely make use of these kinds of 'microscopic' descriptions.

Once more we return to the example of water. Without knowing the exact details of the phase the water molecules are in, we can classify water by its *symmetries*. Let us first have a look at the case where the material forms a liquid. All the molecules move around, they collide and each go their own way again. This is a situation with very little structure. When we consider the liquid at a certain point in space and we compare that with a position farther along, we will not see any difference. This is what we call a symmetry of the system and in this particular case it is a translational symmetry. How does this compare to ice? The molecules form a crystal and no longer move around, they are fixed to a grid. Now when we look at a certain point in space, for instance where a water molecule is located, we will not find the same situation when we translate to a different position in space. It is only a symmetry when translating by steps equal to the distance between the water molecules. In the case of ice there is more structure and therefore less symmetry. The *continuous* translation symmetry of the liquid is broken to a *discrete* translation symmetry. There is a mathematical theory at the root of describing symmetries. This field is called group theory (6) and is used in all corners of modern physics.

(6) Group theory is a branch of mathematics that studies the properties of groups and classifies them. A group is a set of elements together with an operation that acts on these elements, which have to obey four conditions. This might sound very abstract (it is mathematics after all!), but I will give one specific example of a group, which hopefully makes it more tangible. The set of all integers, which is represented by the symbol $\mathbb{Z}$, together with addition (the operation) is an example of a group. The set $\mathbb{Z}$ is composed of numbers such as $3, 18, 49899, -1, -886$, but excludes numbers with decimal places. By definition a group must meet the following requirements (otherwise it cannot be classified as a group).

Closure: If we take two arbitrary integers their sum always results in another integer, i.e. another element of the group.

Associativity: When adding three integers the order in which we do so does not matter, $(3 + 11) + 22 = 3 + (11 + 22)$.

Identity element: There is one element in the group which acts trivially on all the other elements. In the case of the integers, the identity element is the number 0, since for instance $-3 + 0 = -3$.

Inverse element: For each element in the group, there is an element such that they add to the identity. In our example, the inverse of a number is obtained by acting with a minus sign, i.e. $-11 + 11 = 0$.

Topological phase In this thesis, I investigate materials that are in a different type of phase than those described for water. It concerns phases that are not characterized by a symmetry such as rotation or translation. This type of phase is referred to as a topological phase (7). Until the 1980's it was believed that all phases and phase transitions could be described by the underlying symmetries that were discussed before, until an important discovery was made which is nowadays known as the *quantum Hall effect* (8).

A material which is in a quantum Hall phase is characterized by the quantization of the Hall resistance, which can be expressed as $R_H = \frac{p}{q}\frac{h}{e^2}$, where p and q are integers that denote the specific quantum Hall phase, h is Planck's constant and e denotes the charge of an electron. The numbers h and e are so-called constants of nature, which means that they always have the same value in contrast to, for instance, pressure or temperature.

When considering two materials that are characterized by different values of the Hall resistance (different values of p and q), their symmetries are the same. And yet they have a different resistance, which is a physical property. From this, one must conclude that the classification of phases on the basis of its symmetries is not always sufficient. The reason that these phases are called topological phases is because of the exact quantization of the resistance. It does not matter whether the shape of the material is different, or there are impurities in the sample. As has been explained in (7), the topology of an object is not sensitive to local changes, but it is a feature of the global system. After the discovery of the quantum Hall phases many more topological phases have been predicted in theory and observed in experiments, nonetheless the quantum Hall effect is still an active area of research and is one of the topics treated in this thesis.

(7) Topology is a branch of mathematics which investigates the properties of an object that do not change (remain invariant) under smooth deformations of the object. Not all deformations are permitted, for instance one is not allowed to tear or glue.

A well-known example is a coffee cup and a donut. At first glance these are two different objects, but if we allow deformations such as described above, we may bend and stretch a coffee cup in such a way that it is transformed into a donut. Therefore these objects are topologically equivalent. The topological invariant that characterizes them is the number of holes in the object. The coffee cup and donut both have one hole, but a sphere for instance has no holes and is not topologically equivalent to a donut. Topology is a global property of an object. Locally we can distort the object without changing its topology. The number of holes remains the same regardless of where they are exactly.

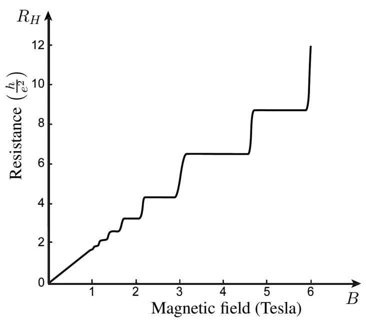
If this kind of equivalence seems unnatural, remember that two women are very different, but we can still choose to classify them as women based on their sex chromosomes. Such a classification can be helpful to distill certain properties as long as we remember what the classification is based on.

Quasiparticles One of the reasons to study topological phases is that in some of these systems particles emerge with special properties. Consider for example the fraction quantum Hall effect. In this system particles can exist with a smaller charge than that of an electron. The astounding thing about this is that this system is built up of electrons, which are elemental particles, so smaller should not be possible. Yet particles with a smaller charge are being observed.

The electrons that form a quantum Hall system behave as if they were a sort of liquid. Locally there can be changes in density, which behave as particles of fractional charge. Considering that the system comprises electrons and not these particles, we speak of quasiparticles.

(8) In (3) the Hall effect was discussed, but this phenomenon is not the whole story. In 1980, a similar experiment was conducted, but now a system was created in which the electrons could move in only two dimensions. This was achieved by placing two suitable materials on top of each other, in such a way that the electrons can only move along the interface between the two materials.

When the system was cooled down to approximately one degree above absolute zero and a very strong magnetic field was applied in the direction perpendicular to the plane in which the electrons can move, it was found that the resistance no longer depended linearly on the strength of the magnetic field. Instead, plateaus arise in the graph, which is depicted in the figure below.



It might be time for a short intermezzo to encourage the reader who has lost track. It does not matter if you do not understand every bit of the text so far. To be able to thoroughly understand this, you need knowledge of many abstract theories, which is preceded by many years of practice in mathematics. What you need to understand at this point is that there exist materials (many particles together) that under certain conditions end up in a phase where unusual physical properties occur. In the case of a material that is in a quantum Hall phase, the resistance becomes quantized and in some cases exotic quasiparticles emerge in the material. I would like to focus somewhat more on these quasiparticles, because they play a big role in my research, but also because they have an exciting application for industry.

Their fractional charge is not the only aspect that makes these quasiparticles special; another property is that they have exotic *statistics*. I will explain this property in the following. Imagine that we have two identical particles, for instance two electrons. Identical means that they are indistinguishable from one another. Now we bring one of the electrons around the other in a full circle, which is schematically depicted in figure 2.1 on page 30. We can ask ourselves the question: can we somehow measure the difference between the situation where the two electrons did not move and the situation where one circulation has been made? With electrons and any other fundamental particle this difference is not measurable, but with the quasiparticles of the quantum Hall phase it is, this is what we call nontrivial statistics. Certain types of quasiparticles have been predicted from theoretical physics (they have not yet been observed with complete certainty) that have special statistics which makes them useful as a kind of hardware for a topological quantum computer (9).

My thesis In the last part of this summary I would like to summarize what is discussed in the separate chapters of my thesis. Chapters 1 and 2 serve as an introduction and discuss the existing knowledge concerning topological phases and quantum Hall systems. In chapter 1 another field is discussed as well, that has not yet been mentioned in this summary, namely cold atom systems. This is a fairly new field and it can be used as a simulator of, for example, quantum Hall materials. By cleverly choosing an array of lasers, a cloud of cold atoms can be made to behave as if it is a quantum Hall system. Chapter 2 concentrates mainly on discussing phase transitions between different topological phases and how such transitions can be described using the specific quasiparticles that exist in that phase.

In chapter 3 a system is discussed that has a charged particle moving in three dimensions, while subject to a magnetic field. This configuration is proposed as a candidate for a new kind of topological phase [105]. We (Sander Bais, Kareljan Schoutens and I) have determined the symmetry of this system and have used group theory to find the energy levels of the particle. A similar issue is discussed in chapter 4. The difference there is that the particle can only move in two dimensions. We (Benoit Estienne, Kareljan Schoutens and I) consider a particle confined to the plane, but also a particle moving on the surface of a sphere.

Chapters 5 and 6 discuss phase transitions between different topological phases. We (Sander Bais, Joost Slingerland and I) expand the existing theory and discuss specific examples of these kinds of processes. Furthermore we look at what happens when two

(9) Computer parts are getting smaller and smaller. As this trend continues we will automatically reach a regime where classic laws of physics no longer apply and the computer parts start to behave according to the laws of quantum mechanics (4). At first glance this seems to be a problem, but in the 1980's a number of physicists realized that this also brings forth great new possibilities. A computer that uses quantum mechanics is called a quantum computer and is (still only in theory) capable of solving specific problems much faster. An example is the factorization of prime numbers, which is nowadays being used for the encryption of information so that it can be sent safely. Quantum computers would make this way of encryption useless because they can easily crack it.

In short, a classic computer uses *bits* where the state a bit is in can be indicated by either 0 or 1. These bits can be used to store and process information. A quantum computer uses *qubits* and the fundamental difference is that a qubit can be either 0 or 1, but also 0 and 1 at the same time. This phenomenon is called superposition and is very common in quantum mechanics. Only when we measure which state the qubit is in do we find 0 or 1, until that time we can only talk about the probability of finding 0 or 1. This phenomenon has practical uses: considering calculations can now be done parallel.

A disadvantage of the quantum computer is that it is highly sensitive to its surroundings, which can cause errors in storing information or computing. A solution to this problem could be a topological quantum computer. The qubits of this type of computer are formed by the quasiparticles that exist in a certain topological phase. These particles have the disposition of being insensitive to the influences of their surroundings. If their position changes a bit or the shape of the material that carries them changes, the information is still stored, as the topology of the system is insensitive to these kinds of local changes (7).

systems that carry different topological phases are adjacent to each other. We derive what occurs at the boundary between the two phases and which quasiparticles can exist there. One of the examples is a phase with the sort of quasiparticles which are useful for a topological quantum computer.